Statistical Description of Multipath Fading

- The basic Rayleigh or Rician model gives the PDF of envelope
- But: how fast does the signal fade?
- How wide in bandwidth are fades?

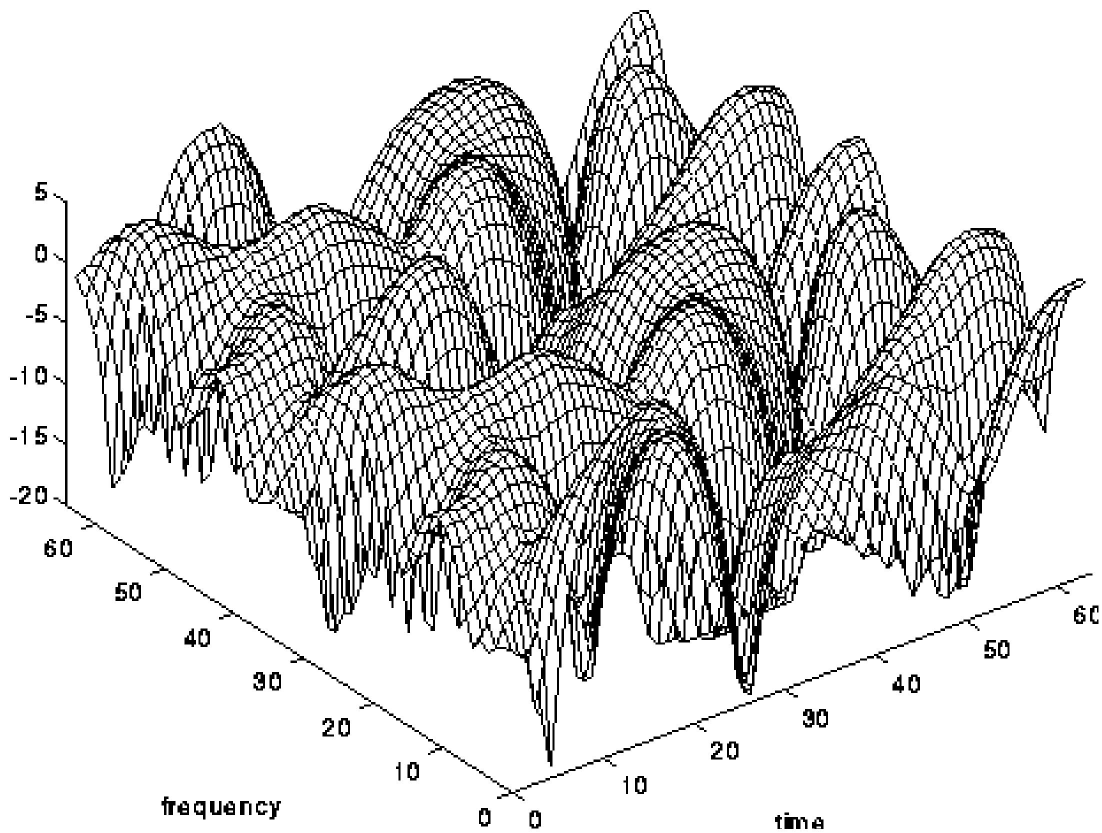
Typical system engineering questions:

- What is an appropriate packet duration, to avoid fades?
- For frequency diversity, how far should one separate carriers?
- How far should one separate antennas for diversity?
- What is good a interleaving depth?
- What bit rates work well?
- Why can't I connect an ordinary modem to a cellular phone?

The models discussed in the following sheets will provide insight in these issues

The Mobile Radio Propagation Channel

A wireless channel exhibits severe fluctuations for small displacements of the antenna or small carrier frequency offsets.



Channel amplitude in dB versus location (= time * velocity) and frequency

Multipath fading is characterized by two distinct mechanisms

1. Time dispersion

- Time variations of the channel are caused by motion of the antenna
- Channel changes every half a wavelength
- Moving antenna gives Doppler spread
- Fast fading requires short packet durations, thus high bit rates
- Time dispersion poses requirements on synchronization and rate of convergence of channel estimation
- Interleaving may help to avoid burst errors

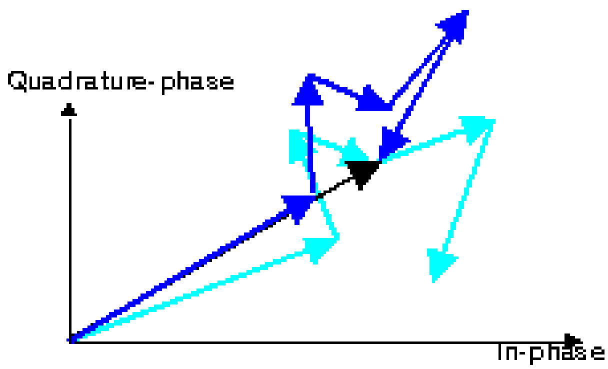
2. Frequency dispersion

- Delayed reflections cause intersymbol interference
- Channel Equalization may be needed.
- Frequency selective fading
- Multipath delay spreads require long symbol times
- Frequency diversity or spread spectrum may help

Narrowband signal (single frequency)

- Transmit: $\cos(2\pi\, f_c\, t)$
- Receive: $I(t)\cos(2\pi\, f_c\, t) + Q(t)\cos(2\pi\, f_c\, t)$

$$= R(t)\cos(2\pi\, f_c\, t + \phi)$$



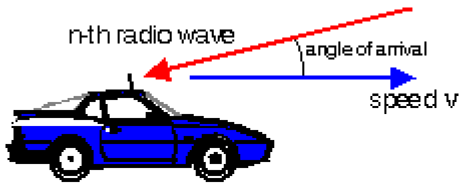
I-Q phase trajectory

- As a function of time, $I(t)$ and $Q(t)$ follow a random trajectory through the complex plane

- Intuitive conclusion: Deep amplitude fades coincide with large phase rotations

Doppler shift

- All reflected waves arrive from a different angle
- All waves have a different Doppler shift



The Doppler shift of a particular wave is

$$f_0 = \frac{v}{c} f_c \cos\phi$$

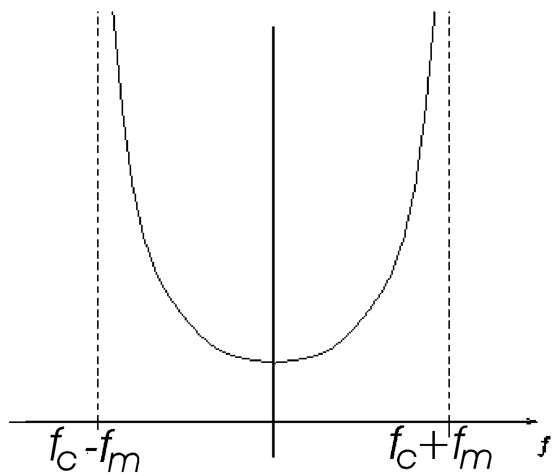
- Maximum Doppler shift: $f_D = f_c\, v/c$

Joint Signal Model

- Infinite number of waves
- Uniform distribution of angle of arrival ϕ: $f_\Phi(\phi) = {}^1/_{2\pi}$
- First find distribution of angle of arrival the compute distribution of Doppler shifts
- Line spectrum goes into continuous spectrum

Doppler Spectrum

If one transmits a sinusoid, what are the frequency components in the received signal?



- Power density spectrum versus received frequency
- Probability density of Doppler shift versus received frequency

- The Doppler spectrum has a characteristic U-shape.
- Note the similarity with sampling a randomly-phased sinusoid
- No components fall outside interval [f_c- f_D, f_c+ f_D]
- Components of + f_D or -f_D appear relatively often
  - Fades are not entirely "memory-less"

Derivation of Doppler Spectrum

The power spectrum *S(f)* is found from

$$S(f_0) = \bar{p}\left[f_\Phi(\phi)G(\phi) + f_\Phi(-\phi)G(-\phi) \right] \left| \frac{d\phi}{df} \right|_{f_0}$$

where

$f_\Phi(\phi) = 1/(2\pi)$ is the PDF of angle of incidence

$G(\phi)$ the antenna gain in direction ϕ

$\bar{p}$ local-mean received power

and

$$f_0 = f_c \left(1 + \frac{v}{c} \cos\phi \right)$$

One finds

$$\left| \frac{d\phi}{df} \right| = \frac{1}{\sqrt{f_D^2 - (f - f_c)^2}}$$

Vertical Dipole

- A vertical dipole is omni-directional in horizontal plane
- $G(\phi) = 1.5$

We assume

- Uniform angle of arrival of reflections
- No dominant wave

Receiver Power Spectrum

$$S(f) = \bar{p}\,\frac{3}{2\pi}\,\frac{1}{\sqrt{f_D^2 - (f - f_c)^2}}$$

- Doppler spectrum is centered around f_c
- Doppler spectrum has width $2\,f_D$

Magnetic loop antenna

- $G(\phi) = 1.5 \sin^2(\phi - \phi_0)$

  with ϕ_0 the direction angle of the antenna
- This antenna does not see waves from particular directions:

  it removes some portion of the spectrum

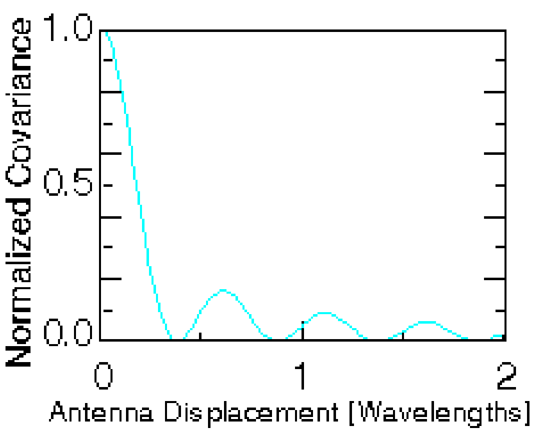
Autocorrelation of the signal

- We know the Doppler spectrum.

  But how fast does the channel change?

Wiener-Khinchine Theorem

- Power density spectrum of a random signal is the Fourier Transform of its autocorrelation
- Inverse Fourier Transform of Doppler spectrum gives autocorrelation of *I(t)* and *Q(t)*



Auto-covariance of received signal amplitude $R^2 = I^2 + Q^2$

Derived from autocorrelation of *I* and *Q*.

Derivation of Autocorrelation of *IQ*-components

We define the autocorrelation

$$g(\tau) \;=\; \mathrm{E}\; I(t)I(t+\tau) \;=\; \mathrm{E}\; Q(t)Q(t+\tau)$$

$$= \int_{f_c - f_D}^{f_c + f_D} S(f)\; \cos 2\pi(f - f_c)\tau\, df$$

So

- Autocorrelation depends on $S(f)$, thus on distribution of angles of arrival
- $g(\tau = 0)$ = local-mean power: $g(0) = \mathrm{E}\; I^2(t) = \bar{p}$

Note that

- In-phase component and its derivative are independent

$$g'(\tau) \;=\; \mathrm{E}\; I(t)\frac{dI}{dt} \;=\; 0$$

For uniform angle of arrival

The autocorrelation function is

$$g(\tau) = \mathrm{E}\; I(t)I(t+\tau) = \bar{p}\; J_0\left(2\pi f_D \tau\right)$$

where

J_0 is zero-order Bessel function of first kind

f_D is the maximum Doppler shift

τ is the time difference

Note that the correlation is a function of distance or time offset:

$$f_D \tau = f_c \frac{v}{c} \tau = \frac{d}{\lambda}$$

where

d is the antenna displacement during τ, with $d = v\,\tau$

λ is the carrier wavelength (30 cm at 1 GHz)

Relation between I and Q phase

S.O. Rice: Cross-correlation

$$h(\tau) = \mathrm{E}\; I(t)Q(t+\tau) = -\mathrm{E}\; Q(t)I(t+\tau)$$

$$= \int_{f_c - f_D}^{f_c + f_D} S(f)\; \sin 2\pi(f - f_c)\tau \, df$$

For uniformly distributed angle of arrival ϕ

- Doppler spectrum $S(f)$ is even around f_c
- Crosscorrelation $h(\tau)$ is zero for all τ
- $I(t_1)$ and $Q(t_2)$ are independent

For any distribution of angles of arrival

- $I(t_1)$ and $Q(t_1)$ are independent $\{\tau = 0$: $h(0) = 0\}$

Autocorrelation of amplitude $R^2 = I^2 + Q^2$

Derivation of E$R(t)R(t+\tau)$

Davenport & Root showed that

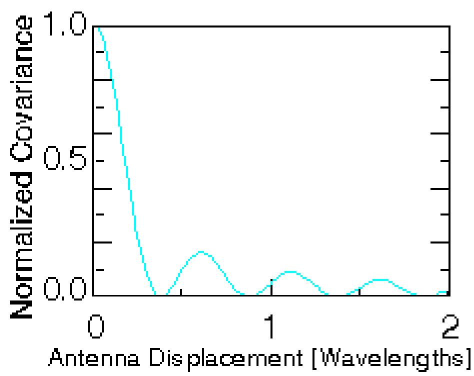
$$\mathrm{E}\, R(t)R(t+\tau) = \frac{\pi}{2}\bar{p} \; F\left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{g^2(\tau)+h^2(\tau)}{\bar{p}^2}\right]$$

where F[a, b, c ; d] is the hypergeometric function

Using a second order series expansion of F[a, b, c ; d]:

$$\mathrm{E}\, R(t)R(t+\tau) = \frac{\pi}{2}\bar{p}\left[1 + \frac{1}{4}\frac{g^2(\tau)}{\bar{p}^2}\right]$$

Result for Autocovariance of Amplitude

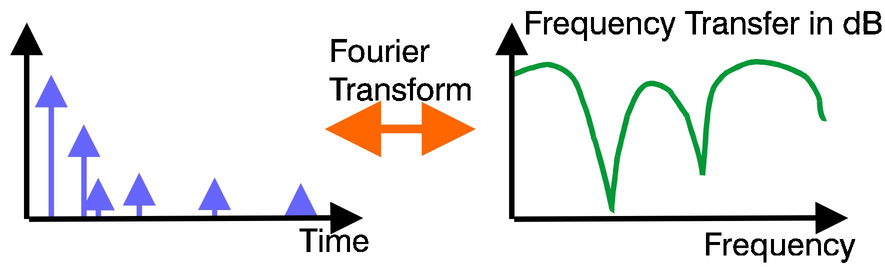


- Remove mean-value and normalize
- Autocovariance

$$C = J_0^2(2\pi f_D \tau)$$

Delay Profile

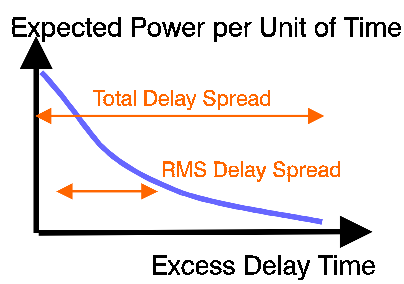
Typical sample of impulse response *h(t)*



If we transmit a pulse $\delta(t)$ we receive $h(t)$

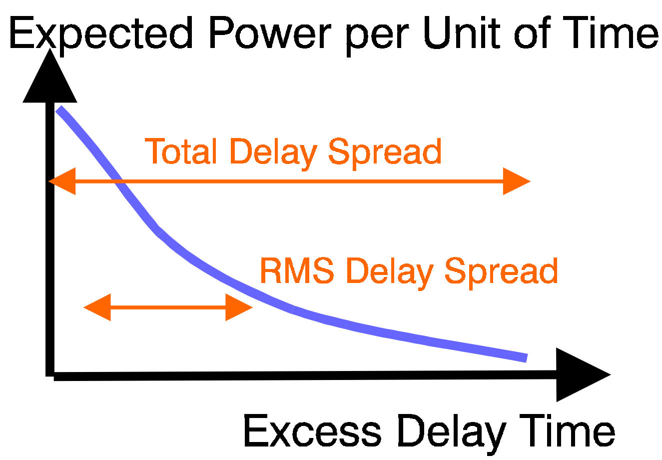
Delay profile:

PDF of received power: "average $h^2(t)$"



Local-mean power in delay bin $\Delta\tau$ is $\bar{p}\ f_\tau(\tau)\Delta\tau$

RMS Delay Spread and Maximum delay spread



Definitions

n-th moment of delay spread

$$\mu_n = \int_0^{\infty} \tau^n \, f_\tau(\tau) \, d\tau$$

RMS value

$$T_{RMS} = \frac{1}{\mu_0^2} \sqrt{\mu_0 \mu_2 - \mu_1^2}$$

Typical Delay Spreads

Macrocells $T_{\mathrm{RMS}} < 8$ µsec

- GSM (256 kbit/s) uses an equalizer
- IS-54 (48 kbit/s): no equalizer
- In mountainous regions delays of 8 µsec and more occur

  GSM has some problems in Switzerland

Microcells $T_{\mathrm{RMS}} < 2$ µsec

- Low antennas (below tops of buildings)

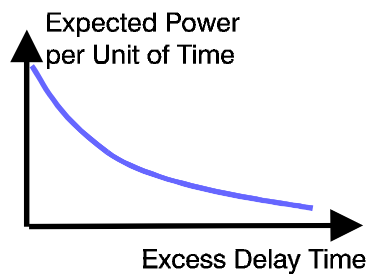
Picocells $T_{\mathrm{RMS}} < 50$ nsec - 300 nsec

- Indoor: often 50 nsec is assumed
- DECT (1 Mbit/s) works well up to 90 nsec

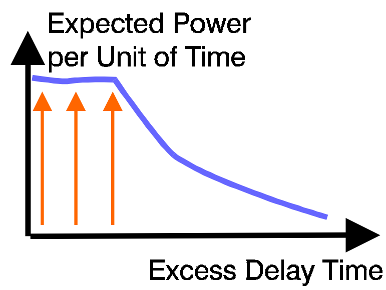
  Outdoors, DECT has problem if range > 200 .. 500 m

Typical Delay Profiles

1) Exponential

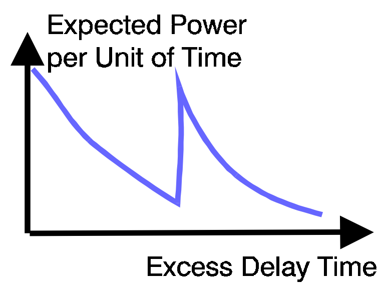


2) Uniform Delay Profile



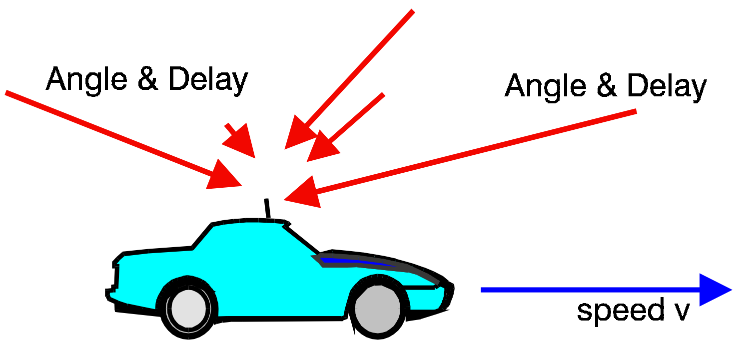
- Experienced on some indoor channels
- Often approximated by *N*-Ray Channel

3) Bad Urban



Effect of Location and Frequency

Model: Each wave has its own angle and excess delay



- Antenna motion changes phase
- changing carrier frequency changes phase

The scattering environment is defined by

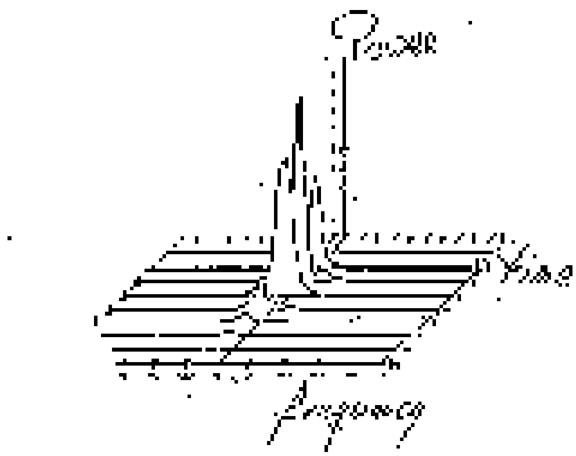
- angles of arrival
- excess delays in each path
- power of each path

Scatter function of a Multipath Mobile Channel

- Gives power as function of

  Doppler Shift (derived from angle ϕ)

  Excess Delay

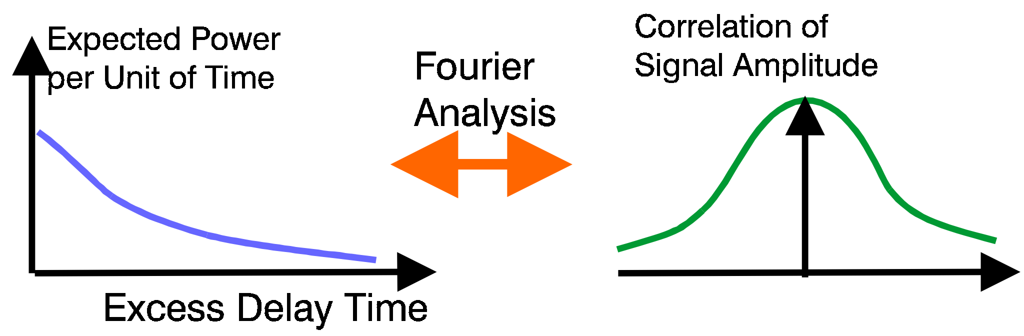


Example of a scatter plot

Horizontal axes:
- x-axis: Excess delay time
- y-axis: Doppler shift

Vertical axis
- z-axis: received power

Correlation of fading vs. Frequency Separation



- When do we experience frequency-selective fading?
- How to choose a good bit rate?
- Where is frequency diversity effective?

In the next slides, we will ...

- give a model for *I* and *Q*, for two sinusoids with time and frequency offset,
- derive the covariance matrix for *I* and *Q*,
- derive the correlation of envelope *R*,
- give the result for the autocovariance of *R*, and
- define the coherence bandwidth.

Inphase and Quadrature-Phase Components

Consider two (random) sinusoidal signals

- Sample 1 at frequency f_1 at time t_1
- Sample 2 at frequency f_2 at time t_2

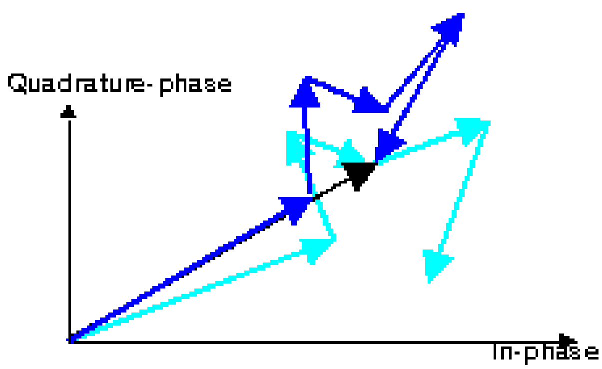
Effect of displacement on each phasor:

Spatial or temporal displacement:

- Phase difference due to Doppler

Spectral displacement

- Phase difference due to excess delay



Mathematical Treatment:

- $I(t)$ and $Q(t)$ are jointly Gaussian random processes
- (I_1, Q_1, I_2, Q_2) is a jointly Gaussian random vector

Covariance matrix of (I_1, Q_1, I_2, Q_2)

$$\Gamma_{I_1,Q_1,I_2,Q_2} = \begin{bmatrix} \bar{p} & 0 & \mu_1 & \mu_2 \\ 0 & \bar{p} & -\mu_2 & \mu_1 \\ \mu_1 & -\mu_2 & \bar{p} & 0 \\ \mu_2 & \mu_1 & 0 & \bar{p} \end{bmatrix}$$

with

$$\mu_1 = \mathrm{E}\, I_1 I_2 = \bar{p}\, \frac{J_0(2\pi \frac{v}{\lambda} \tau)}{1 + 4\pi^2 (f_1 - f_2)^2 T_{RMS}{}^2}$$

and

$$\mu_2 = \mathrm{E}\, I_1 Q_2 = -2\pi(f_2 - f_1)\bar{p}\, \frac{J_0(2\pi \frac{v}{\lambda} \tau)}{1 + 4\pi^2 (f_1 - f_2)^2 T_{RMS}{}^2}$$

where

J_0 is the Bessel function of first kind of order 0.
T_{RMS} is the rms delay spread

Derivation of Joint Probability Density R_1, R_2

- Amplitude $R_1^2 = I_1^2 + Q_1^2$
- Conversion from (I_1, Q_1, I_2, Q_2) to $(R_1, \phi_1, R_2, \phi_2)$.
  - Jacobian is $|J| = R_1 R_2$
  - so the PDF $f(r_1, \phi_1, r_2, \phi_2)$ is $r_1 r_2\, f(i_1 = r_1 \cos\phi_1,\ q_1 = r_1 \sin\phi_1,\ i_2 = r_2 \cos\phi_2,\ q_2 = r_2 \sin\phi_2)$.

Integrating over phases ϕ_1 and ϕ_2 gives

$$f(r_1, r_2) = \frac{r_1 r_2}{\bar{p}^2 (1 - \rho^2)} \exp\left\{ - \frac{r_1^2 + r_2^2}{2\bar{p}(1 - \rho^2)} \right\} I_0 \left(\frac{r_1 r_2 \rho}{\bar{p}(1 - \rho^2)} \right)$$

where

- the Bessel function I_0 occurs due to $\int \exp\{\cos\phi\}\, d\phi$
- the normalized correlation coefficient ρ is

$$\rho^2 = \frac{\mu_1^2 + \mu_2^2}{\bar{p}^2} = \frac{J_0^2 (2\pi \frac{v}{\lambda} \tau)}{1 + 4\pi^2 (f_1 - f_2)^2 T_{RMS}{}^2}$$

Derivation of Envelope Correlation $\mathrm{E}R_1R_2$

Definition:

$$\mathrm{E}\,R_1R_2 \;=\; \int_0^\infty\!\int_0^\infty r_1 r_2\, f(r_1, r_2)\; dr_1 dr_2$$

Inserting the PDF (with Bessel function) gives the Hypergeometric integral

$$\mathrm{E}\,R_1R_2 \;=\; \frac{\pi}{2}\;\bar{p}\,F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \rho^2\right)$$

This integral can be expanded as

$$\mathrm{E}\,R_1R_2 \;=\; \frac{\pi}{2}\;\bar{p}\left[1 \;+\; 2^{-2}\rho^2 \;+\; 2^{-6}\rho^4 \;+\; 2^{-9}\rho^6 \;+\ldots\right]$$

Mostly, only the first two terms are considered

Normalized Envelope Covariance

Definition:

- Correlation coefficient: Normalized covariance $0 \leq C \leq 1$

$$C_{R_1 R_2} = \frac{\mathrm{COV}(r_1, r_2)}{\mathrm{SIG}(r_1)\ \mathrm{SIG}(r_2)} = \frac{\mathrm{E}\, r_1 r_2 - \mathrm{E}\, r_1 \mathrm{E}\, r_2}{\sqrt{\mathrm{E}\, r_1^2 - \mathrm{E}^2 r_1}\ \sqrt{\mathrm{E}\, r_2^2 - \mathrm{E}^2 r_2}} =$$

where

- Local-mean value: $\mathrm{E}R_1 = \sqrt{(\pi p \ / 2)}$
- Variance: $\mathrm{VAR}R_1 = \mathrm{SIG}^2 R_1 = (2 - \pi / 2)p$
- Correlation: $\mathrm{E}R_1 R_2 \approx \pi p \ /2\ [1 + \rho^2 / 4]$

Result

Thus, after some algebra,

$$C_{R_1, R_2} \approx \rho^2 = \frac{J_0^2\left(2\pi \frac{v}{\lambda} \tau\right)}{1 + 4\pi^2 (f_1 - f_2)^2 T_{RMS}{}^2}$$

Special cases

- Zero displacement / motion: $\tau = 0$
- Zero frequency separation: $\Delta f = 0$

Coherence Bandwidth

Definition of Coherence Bandwidth:

- Coherence. Bandwidth is the frequency separation for which the correlation coefficient is down from 1 to 0.5
- Thus $1 = 2\pi(f_1 - f_2)\, T_{RMS}$

  so Coherence Bandwidth BW = 1 ($2\pi\, T_{RMS}$)
- We derived this for an exponential delay profile

Another rule of thumb:

- ISI affects BER if $T_b > 0.1T_{RMS}$

Conclusion:

- Either keep transmission bandwidth much samller than the coherence bandwidth of the channel, or
- use signal processing to overcome ISI, e.g.
  - Equalization
  - DS-CDMA with rake
  - OFDM

Duration of Fades

In the next slides we study the temporal behavior of fades.

Outline:

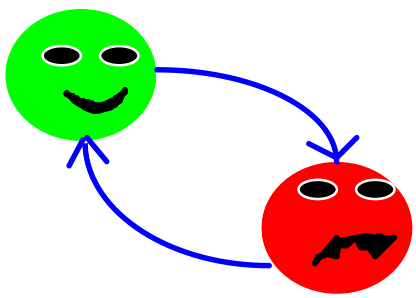
- # of level crossings per second
  - Model for level crossings
  - Derivation
    - Model for I, Q and derivatives
    - Model for amplitude and derivative
  - Discussion of result
- Average non-fade duration
- Effective throughput and optimum packet length
- Average fade-duration

Gilbert-Elliot Model

Very simple mode: channel has two-states

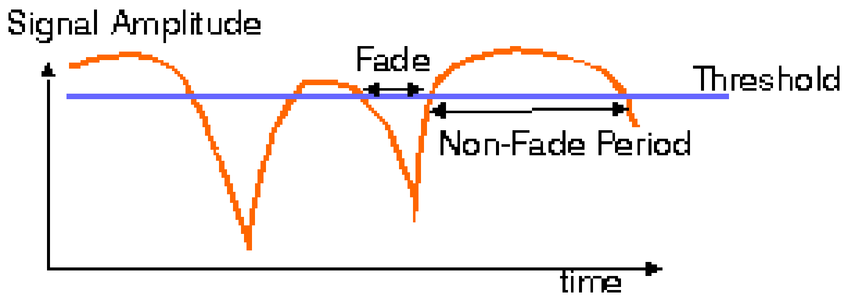
- Good state: Signal above “threshold”, BER is virtually zero
- Poor state: “Signal outage”, BER is 1/2, receiver falls out of sync, etc

Markov model approach:
- Memory-less transitions
- Exponential distribution of sojourn time

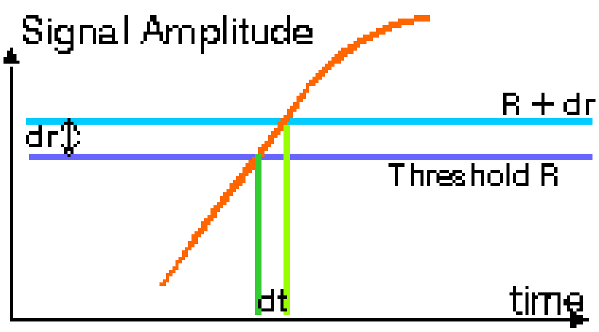


This model may be sufficiently realistic for many investigations

Level crossings per second



- Av. number of crossings per sec = [av. interfade time]$^{-1}$



Number of level crossing per sec is proportional to

- speed *r*' of crossing *R* (derivative *r*' = *dr*/*dt*)
- probability of *r* being in [*R*, *R* + *dR*]

Derivation of Level Crossings per Second

The expected number of crossings in positive direction per second is [Rice]

$$N^{+} = \int_{0}^{\infty} \hat{r}\, f(\hat{r}, R)\, d\hat{r}$$

- Random process r^ is derivative of the envelope r w.r.t. time
- Note: here we need the *joint* PDF;
  not the *conditional* PDF $f(r\text{^} \mid r=R)$

Covariance matrix of (*I*, *Q*, *I*^, *Q*^)

In-phase, quadrature and the derivatives are Jointly Gaussian with

$$\Gamma_{I,Q,\hat{I},\hat{Q}} = \begin{bmatrix} \bar{p} & 0 & 0 & b_1 \\ 0 & \bar{p} & -b_1 & 0 \\ 0 & -b_1 & b_2 & 0 \\ b_1 & 0 & 0 & b_2 \end{bmatrix}$$

where

b_n is n-th moment of Doppler spectral power density

$$b_n = (2\pi)^n \int_{f_c - f_D}^{f_c + f_D} S(f)\,(f - f_c)^n \, df$$

For uniform angles of arrival

$$b_n = \begin{cases} \bar{p}\,(2\pi f_D)^n \dfrac{(n-1)!!}{n!} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

where $(n\text{-}1)!! = 1\ 3\ 5\ .....(n\text{-}1)$

Joint PDF of R, R^, Φ, Φ^

.... After some algebra ...

$$f(r,\hat{r},\phi,\hat{\phi}) = \frac{r^2}{4\pi^2 \overline{p} b_2} \exp\left\{ -\frac{1}{2}\left(\frac{r^2}{\overline{p}} + \frac{\hat{r}^2}{b_2} + \frac{r^2 \hat{\phi}^2}{b_2} \right) \right\}$$

So

- *r* and *r*^ are independent
- phase ϕ is uniform

Averaging over ϕ and ϕ^ gives

- *r* is Rayleigh
- *r*^ is zero mean Gaussian with variance b_2

Level Crossings per Second

We insert this pdf in

$$N^+ = \int_0^\infty \hat{r}\, f(\hat{r}, R = R_0)\, d\hat{r}$$

We find

$$N^+ = \frac{\sqrt{2\pi}\; f_D}{\sqrt{\eta}}\; e^{-\frac{1}{\eta}}$$

where

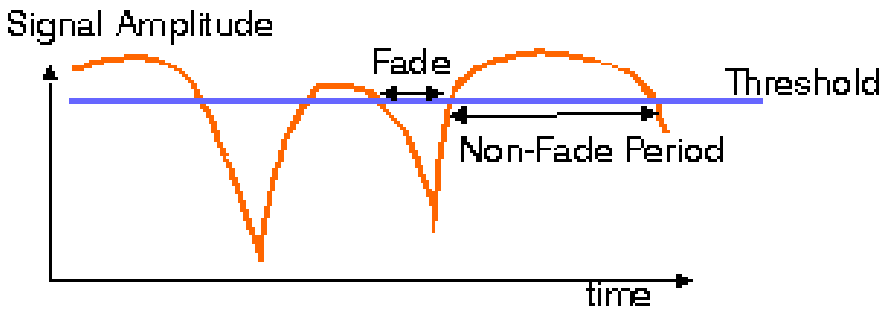
η is the fade margin $\eta = 2\,p\,/\,(R_0^2)$

Level crossing rate has a maximum for thresholds R_0 close to the mean value of amplitude

Similarly for Rician fading

$$N^+ = \sqrt{\pi \bar{p}}\; f_D\; f_R(R_0)$$

Average Fade / Nonfade Duration



- Fade durations are relevant to choose packet duration
- Duration depend on
  - fade margin $\gamma = 2\,p / R_0^2$
  - Doppler spread

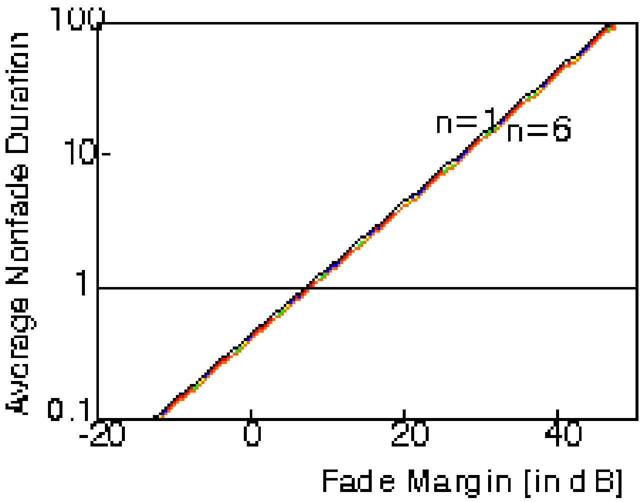
Average fade duration T_F

$$N^- \; T_F \;=\; \Pr(R \leq R_0)$$

Average nonfade duration T_{NF}

$$N^+ \; T_{NF} \;=\; \Pr(R \geq R_0)$$

Average nonfade duration



$$T_{NF} = \frac{e^{-\frac{1}{\eta}}}{\frac{\sqrt{2\pi}\; f_D}{\sqrt{\eta}}\; e^{-\frac{1}{\eta}}} = \frac{\sqrt{\eta}}{\sqrt{2\pi}\; f_D}$$

- Average nonfade duration is inversely proportional to Doppler spread
- Average nonfade duration is proportional to fade margin

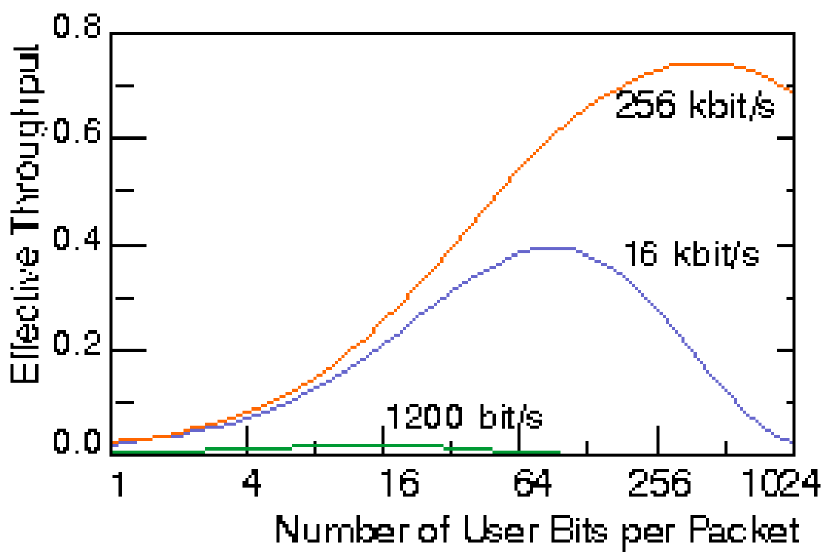
Optimal Packet length

We want to optimize

$$\text{Effective throughput} = \frac{\#\text{User bits}}{\#\text{Packet bits}} \text{ Probability of success}$$

This gives a trade off between

- Short packets: much overhead (headers, sync. words etc).
- Long packets: may experience fade before end of packet.



- At 1200 bit/s, throughput is virtually zero: Almost all packets run into a fade. Packets are too long
- At high bit rates, many packets can be exchanged during non-fade periods. Intuition:

Packet duration < < Av. nonfade duration

Derivation of Optimal Packet length

- Assume exponential, memoryless nonfade durations
  (This is an approximation: In reality many nonfade periods have duration of $\lambda/2$, due to U-shaped Doppler spectrum)

- Successful reception if
  1) Above threshold at start of packet
  2) No fade starts before packet ends

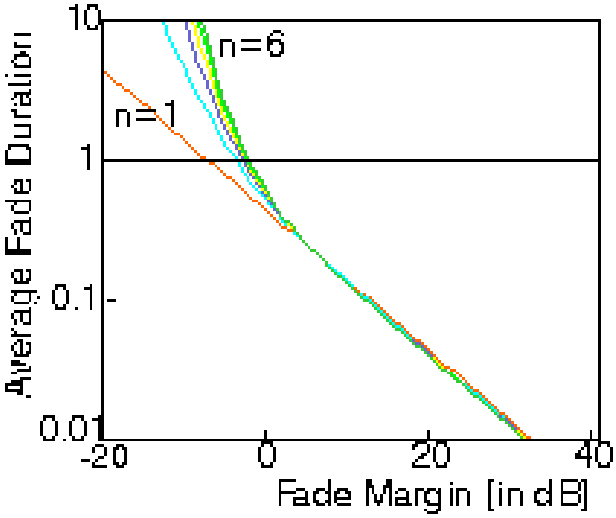
Formula:

$$P(\text{succ}) = \exp\left(-\frac{1}{\eta} - \frac{T_L}{T_{NF}}\right) = \exp\left(-\frac{1}{\eta} - \frac{\sqrt{2\pi}\, f_D T_L}{\sqrt{\eta}}\right)$$

with T_L packet duration

- **large fade margin: second term dominates:**
  - performance improves only slowly with increasing η
  - Outage probability is too optimistic

Average fade duration



$$T_F = \frac{\sqrt{\eta}}{\sqrt{2\pi}\ f_D}\ [\exp(\eta) - 1]$$

- Average fade / nonfade duration is inversely proportional to Doppler spread
- At very low fading margins, effect of the number of interfering signals *n* is significant
- Fade durations rapidly reduce with increasing margin, but time between fades increases much slower
- Experiments: For Large fade margins: exponentially distributed fade durations
- Relevant to find length of error bursts and design of interleaving