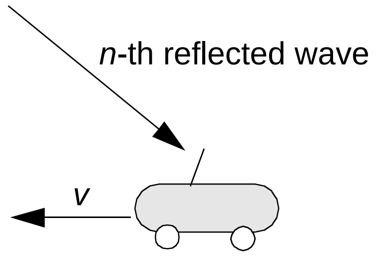
Multipath fading

If the mobile antenna receives a large number, say N, reflected and scattered waves, the instantaneous received power becomes a random variable, dependent on the location of the antenna. Initially, we address the case of an unmodulated carrier, thus the transmitted signal has the form $v(t) = \cos(\omega_c t + \psi)$. Let the nth wave with amplitude c_n and phase ϕ_n arrive from an angle α_n relative to the direction of the motion of the antenna.



The Doppler shift Δf_n of this wave is

$$\Delta f_n = \frac{v}{\lambda} \cos \alpha_n ,$$

where v is the speed of the antenna. The received signal $r(t)$ can be expressed as

$$r(t) = \sum_{n=1} c_n \cos(2\pi f_c t + \psi + \phi_n + 2\pi \Delta f_n t)$$

An inphase-quadrature representation of the form $r(t) = I(t) \cos \omega_c t - Q(t) \sin \omega_c t$ can be found with

$$I(t) = \sum_{n=1} c_n \cos\left(\frac{2\pi v f_c t}{c} \cos \alpha_n + \psi + \phi_n\right)$$

and

$$Q(t) = \sum_{n=1}^{N} c_n \sin\left(\frac{2\pi v f_c t}{c}\cos\alpha_n + \psi + \phi_n\right) \quad .$$

PDF of signal amplitude

In order to obtain the pdf of the signal amplitude ρ we observe the random processes $I(t)$ and $Q(t)$ at one particular instant t_o. If $N \to \infty$ and the terms in the summings are i.i.d., the central limit theorem says that $I(t_o)$ and $Q(t_o)$ are (zero-mean) Gaussian. The received signal $r(t) = \rho(t)\cos(2\pi f_c t + \theta(t))$ has a Rayleigh amplitude $\rho(t)$, with $\rho(t) = \sqrt{I^2(t) + Q^2(t)}$ and a uniform phase $\theta(t)$ with $\theta(t) \in (0, 2\pi)$. The pdf of $\rho(t)$ is

$$f_\rho(\rho) = \frac{\rho}{\sigma} exp\left(-\frac{\rho^2}{2\sigma^2}\right)$$

where σ^2 is the variance of $I(t)$ and $Q(t)$.

<u>Exercise</u> Let $z = x + jy$ with x and y i.i.d. Gaussian with zero mean and variance σ^2. Show that $\rho = |z|$ is Rayleigh, with $\mathrm{E}\rho = \sqrt{\frac{\pi}{2}}\sigma$ and $\mathrm{E}\rho^2 = 2\sigma^2$.

Simulations have shown that the Rayleigh pdf appropriately describes the fading of the amplitude if $N > 6$. Measurements over non-line-of-sight paths at UHF frequencies in urban environments confirmed the accuracy of the Rayleigh pdf.

The instantaneous power p, with $p = \frac{1}{2}\rho^2$,thus, averaged over one RF-cycle, has the exponential pdf

$$f_p(p) = f_\rho(\sqrt{2\rho})\left|\frac{d\rho}{dp}\right| = \frac{1}{\sigma^2}exp\left(-\frac{p}{\sigma^2}\right)$$

where $\sigma^2 = \text{'E}p = \bar{p}$ is the local-mean power.

<u>Exercise</u> Find the cumulative distribution of ρ. In a cellular voice channel, a signal outage occurs if the instantaneous signal-to-noise ratio $\frac{\rho}{N_\rho}$ is less than $z = 10$ (10 dB). Find the required local-mean power to ensure an outage probability of less than 1%. Find the corresponding fade margin, defined as the excess power above the threshold zN_p .

If the set of reflected waves are dominated by one strong component with amplitude c_o, Rician fading is a more appropriate model. Without lack of generality we take $\phi_o = 0$, so the in-phase and quadrature component I and Q have jointly Gaussian pdf

$$f_{I,Q}(I, Q) = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-(I-c_o)^2 - Q^2}{2\sigma^2}\right).$$

The amplitude $\rho = \sqrt{I^2 + Q^2}$ and the $\text{phase}\theta = \arctan\left(\frac{Q}{I+c_0}\right)$ have the joint pdf

$$f_{\rho,\theta}(\rho, \theta) = \frac{\rho}{2\pi\sigma^2}\exp\left(-\frac{\rho^2 - 2c_o\rho\cos\theta + c_o^2}{2\sigma^2}\right)$$

with σ^2 the local-mean scattered power and $\frac{1}{2}c_o^2$ the power of the dominant component.

The pdf of ρ is found from the integral

$$f_\rho(\rho) = \int_{-\pi}^{\pi} f_{\rho,\theta}(\rho,\theta)\, d\theta$$

$$= \frac{\rho}{\sigma^2}\exp\left(-\frac{\rho^2 + c_o^2}{2\sigma^2}\right) I_o\left(\frac{c_o\rho}{\sigma^2}\right),$$

where I_o is the modified Bessel function of the first kind and zero order, defined as

$$I_o(x) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta} d\theta$$

<u>Exercise</u> Show that the Rician pdf of p turns into a Rayleigh pdf for $c_o \to 0$ and into a Gaussian pdf for $c_o \gg \sigma$. Use the properties of $I_o(\bullet)$ as summarized in Mathematical handbooks, e.g. by Abramowitz.

Expressed in terms of the local-mean power $\bar{p}$ with $\bar{p} = \frac{1}{2}c_o^2 + \sigma^2$ and the Rician K-factor with $K = \frac{c_o^2}{2\sigma^2}$, the pdf of the signal amplitude becomes

$$f_\rho(\rho) = (1+K)\, e^{-K}\frac{\rho}{\bar{p}}\exp\left(-\frac{1+K}{2\bar{p}}\rho^2\right) I_o\left(\sqrt{\frac{2K(1+K)}{\bar{p}}}\rho\right)$$

<u>Exercise</u> Show that for a large local-mean signal-to-noise ratio $\bar{p} \gg p_n$, the probability that the instantaneous power p drops below a noise threshold p_n tends to $(K+1)\, e^{-K}\frac{p}{p_n}$.

Experimental results showed that other pdfs can also accurately model the pdf of the received signal amplitude. An interesting property is that the incoherent (power) sum of m i.i.d. Rayleigh fading signals has a gamma distribution power p_t ($p_t = p_1 + p_2 \ldots + p_n$) since

$$f_{p_t}(p_t) = \bigotimes_{i=1}^{m} \frac{1}{\bar{p}} \exp\left(-\frac{p_i}{\bar{p}}\right) = \frac{1}{\bar{p}\Gamma(m)} \left(\frac{p_t}{\bar{p}}\right)^m \exp\left(-\frac{p}{\bar{p}}\right)$$

where f denotes the convolution operation and G(m) is the gamma function, with G(m+1)=m! The amplitude ρ associated with $p\left(p = \frac{1}{2}\rho^2\right)$ then has the Nakagami m-pdf

$$f_\rho(\rho) = \frac{\rho^{2m-1}}{\Gamma(m)\, 2^{m-1} \bar{p}^m} exp\left(-\frac{\rho^2}{2\bar{p}}\right)$$

Note, however, that the incoherent sum of m signals is not a constant-envelope signal. The Nakagami pdf has been proposed as a good experimental approximation of the pdf of the amplitude of a single fading signal both for UHF mobil communication and for HF ionospheric communication. It can also approximate the Rician pdf and the log-normal pdf for small σ_s, except in the extreme tails. Special cases occur for shape factors of $m = \frac{1}{2}$, 1 and ë•. For $m = \frac{1}{2}$, a one-sided Gaussian pdf of the amplitude $(p > 0)$ is obtained. For $m = 1$, a Rayleigh pdf is recovered and for $m \to \infty$, $f_\rho(\rho) \to \delta\left(\rho - \sqrt{2m\bar{p}}\right)$. This agrees with the fact that the incoherent sum of many independent signals behaves as a band-limited Gaussian signal. The power of such a signal (averaged over a time interval T which is substantially larger than the coherence time of the signal phases but shorter than the coherence time of any channel fading) is constant and equal to mp.

RF power spectrum

We assume that $N \to \infty$ and that the angle of arrival α has the (continous) uniform pdf

$f_\alpha(\alpha) = \frac{1}{2\pi}$. For frequencies in the band $f_c\left(1 - \frac{v}{c}\right) < f_o < f_c\left(1 + \frac{v}{c}\right)$, the power density spectrum of the received signal can be found from

$$S(f_o) = \bar{p}\,[f_\alpha(\alpha)\,G_R(\alpha) + f_\alpha(-\alpha)\,G(-\alpha)]\left|\frac{d\alpha}{df}\right|_{f_o}$$

where p is the local-mean power received by an isotropic antenna and $G_R(\alpha)$ is the antenna gain in direction α. The relation between f_o and α is $f_o = f_c\left(1 + \frac{v}{c}\cos\alpha\right)$, so

$\left|\frac{d\alpha}{d_f}\right|_{f_o} = \frac{1}{\sqrt{(f_o - f_c)^2 - f_D^2}}$, where f_D is the maximum Doppler shift $f_D = \frac{vf_c}{c}$. An electrical dipole antenna receives the electric field in vertical direction $'E_z \text{with} G(\alpha) = 1.5$. So, if an unmodulated carrier is transmitted , the received spectrum is

$$S_{E_z}(f) = \frac{3}{2\pi}\frac{p}{\sqrt{f_D^2 - (f - f_c)^2}}$$

Similiarly, one can derive the spectrum of a magnetic loop in the direction of v, receiving H_y, or perpendicular to v, receiving H_x, with antenna gains $G(\alpha) = \frac{3}{2}(\sin^2\alpha)$ and $\frac{3}{2}\cos^2\alpha$, respectively.

<u>Exercise</u> Find an expression for the spectrum $S_{E_z}(f)$ for Rician fading with K = 10, given the angle of arrival of the dominant wave α_o .