Multipath Propagation II

Properties of the signal envelope

Before we can derive properties of the Rayleigh-fading envelope ρ, with $\rho^2 = I^2 + Q^2$, we need to express correlation functions of the inphase and quadrature components $I(t)$ and $Q(t)$. Rice derived the autocorrelation $g(\tau)$ defined as

$$g(\tau) = \mathrm{E}\, I(t)\, I(t+\tau) = \mathrm{E}\, Q(t)\, Q(t+\tau)$$

$$= \int_{f_C - f_D}^{f_C + f_D} S(f) \cos 2\pi (f - f_c)\, \tau\, df$$

with $S(f)$ the power spectral density of the RF-signal. In the special case of uniform angles of arrival, the cross correlation $h(\tau)$, with

$$h(\tau) = \mathrm{E}\, I(t)\, Q(t+\tau) = -\mathrm{E}\, Q(t)\, I(t+\tau)$$

$$= \int_{f_C - f_D}^{f_C + f_D} S(f) \sin 2\pi (f - f_c)\, \tau\, df ,$$

is identical to zero for E_z, H_x and H_y. The behavior of the correlations at $\tau = 0$ can be obtained from the moments of the power spectrum

$$b_n = (2\pi)^n \int_{f_C - f_D}^{f_C + f_D} S(f)\, (f - f_c)^n df.$$

For E_z, one finds $b_n = 0$ for n is odd and

$$b_n = b_o (2\pi f_D)^n \frac{(n-1)!!}{n!}$$

where $(n-1)!! = 1 \cdot 3 \cdot 5 \ldots (n-1)!$ for n is even. In particular $b_o = 1.5\bar{p} = \frac{3\text{'E}_o^2}{4}$ with p the local-mean power for an isotropic antenna. Hence,

$$\text{'E}I^2 = \text{'E}Q^2 = g(0) = b_{(}$$

$$\text{'E}\frac{dI}{dt}I = g'(0) = 0$$

$$\text{'E}\left(\frac{dI}{dt}\right)^2 = -g''(0) = b_2$$

The autocorrelation $g(\tau)$ is $g(\tau) = b_o J_o(2\pi f_D \tau)$ with J_o the Bessel function of zero order[ ].

Davenport and Root [ ] expressed the autocorrelation of the envelope in terms of a hypergeometric function, namely

$$R_\rho(\tau) = \frac{\pi}{2} b_o F\left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{g^2(\tau) + h^2(\tau)}{b_o^2}\right]$$

For $h(\tau) \equiv 0$, a first order expansion of F in terms of $g(\tau)$ gives $R_\rho(\tau) \approx \frac{\pi}{2}\left[b_o + \frac{1}{4}\frac{g^2(\tau)}{b_o^2}\right]$.

<u>Exercise</u> Verify that this approximation gives $\text{'E}\rho^2(0) \approx \frac{5\pi}{8} b_o$ which is very close to theoretical $2b_o$.

Removing the mean amplitude $\text{'E}\rho = \sqrt{\frac{\pi b_o}{2}}$, the autovariance is found as

$C_\rho(\tau) = R_\rho(\tau) - \text{'E}\rho$ where, for the field components

$$E_z: \qquad 'C_\rho(\tau) = \frac{\pi}{8} b_o J_o^2 (2\pi f_D \tau)$$

$$H_x: \qquad 'C_\rho(\tau) = \frac{\pi}{8} b_{oH} [J_o(2\pi f_D \tau) + J_2(2\pi f_D \tau)]^2$$

$$H_y: \qquad 'C_\rho(\tau) = \frac{\pi}{8} b_{oH} [J_o(2\pi f_D \tau) - J_2(2\pi f_D \tau)]^2$$

with b_{oH} the local-mean output power of a (magnetic) loop antenna $\left(b_{oH} = \frac{3'E_o^2}{8} = \frac{b_o}{2} \right)$.

The power spectral density $S_\rho(f)$ of the envelope is found from the Fourier transform

$$S_\rho(f) = \frac{\pi}{2} b_o \int_{-\infty}^{\infty} R_\rho(\tau)\, 'e^{-j2\pi f\tau} d\tau$$

$$\approx \frac{\pi}{2} b_o \delta_\rho(f) + \frac{\pi}{\delta b_o} \int_{-\infty}^{\infty} (g^2(\tau) + h^2(\tau))\, 'e^{-j2\pi f\tau} d\tau$$

Expressing the autocorrelation $g(\tau)$ and $h(\tau)$ of the inphase and quadrature components in terms of the power spectral density $S(f)$ of the received signal, we find [Jakes] for positive frequencies $(0 \le f \le 2f_m)$

$$S_\rho(f) = \frac{\pi}{2} b_o \delta(f) + \frac{1}{16 b_o} \int_{f_c - f_m}^{f_c \, ' \, f_m - f} S(x)\, S(x+f)\, dx$$

It follows that the spectrum of the envelope contains frequencies up to twice the maximum Doppler shift. A mathematical explanation of this is the convolution in frequency domain resulting from the squaring-operation in expression ( ) for the autocorrelation of the envelope.

<u>Exercise</u> Draw a phasor diagram of two interfering waves. Show how the phase and amplitude of the phasor sum can change rapidly even for modest changes of the individ-

ual components.

For the electric field component 'E$_z$, the spectrum of the envelope is

$$S_\rho(f) = \frac{b_o}{8\pi f_D} K\left(\sqrt{1-\left(\frac{f}{2f_D}\right)^2}\right)$$

with K the elliptic integral of the first kind with $K(m) = \int_0^1 (1-t^2)^{-1/2}(1-mt^2)^{-1/2} dt$ [ ].

Exercise If a dominant wave arrives from angle α_o, the received signal power spectrum is $S(f) = S_o(f) + B\delta(f - f_C - f_D \cos\alpha_o)$. Find the spectrum of the resulting envelope. Explain why the new spectrum is still band limited to $2f_D$, but contains discontinuities or peaks at $f_D \pm f_D \cos\alpha_o$.

Derivatives of amplitude and phase

In a Rician-fading channel with zero line-of-sight amplitude $(c_o = 0)$, the inphase and quadrature component and their derivatives are zero-mean jointly Gaussian. The covariance matrix of I_o, Q_o, $\dot{I}_o$, and $\dot{Q}_o$ is

$$c = \begin{bmatrix} b_o & 0 & 0 & b_1 \\ 0 & b_o & -b_1 & 0 \\ 0 & -b_1 & b_2 & 0 \\ b_1 & 0 & 0 & b_2 \end{bmatrix},$$

where b_n is the nth moment of the Doppler spectrum of the scattered power. The determinant of this matrix is $(b_o b_2 - b_1^2)^2$.

Exercise Find the inverse of c and give the joint pdf of I_o, Q_o, $\dot{I}_o$, and $\dot{Q}_o$.

Rice [ ] expressed the pdf of $\rho, \dot{\rho}, \theta$ and $\dot{\theta}$ by transformation of random variables. For a Rician-fading signal with nonzero line-of-sight component, thus with $c_o \neq 0$, one can write

$$I_o = \rho\cos\theta - c_o$$

$$Q_o = \rho\sin\theta$$

$$\dot{I}_o = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta$$

$$\text{and}\ \dot{Q}_o = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta\ .$$

Hence, $dI_o dQ_o d\dot{I}_o d\dot{Q}_o = \rho^2 d\rho d\dot{\rho} d\theta d\dot{\theta}$. After some algebraic operation, this leads to

$$f\left(\rho, \dot{\rho}, \theta, \dot{\theta}\right) = \frac{\rho^2}{4\pi\,(b_o b_2 - b^2{}_1)} \exp - \{ \frac{1}{2b_o b_2 - 2b^2{}_1} [b_2\,(\rho^2 - 2c_o\rho\cos\theta$$

$$+ Q^2\ ) + b_o\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2\right) - 2b_1\rho^2\dot{\theta} + 2b_1 c_o\left(\dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta\right)]\ \}\ .$$

This general result is relevant to a number of more specific properties to be derived later.

<u>Exercise</u> Find $f\left(\rho, \dot{\rho}\right)$ for Rayleigh-fading $(c_o = 0)$. Give an intuitive explanation why $\dot{\rho}$ is normal. Find the variance of $\dot{\rho}$.

Threshold crossing rate

The level crossing rate M_{ρ_o} is defined as the expected number of times that the envelope ρ crosses in positive direction a particular level ρ_o during one second. Given a particular derivative $\dot{\rho}_o\left(\dot{\rho}_o > 0\right)$ of the envelope, the duration Δt of a transition through the range $\rho_o \leq \rho \leq \rho_o + \Delta\rho$ is $\Delta t = \frac{\Delta\rho}{\dot{\rho}_o}$. The conditional expectation of the number of crossings per second is

$$'E\left[M_{\rho_o} \middle| \dot{\rho}_o\right] = \frac{1}{\Delta t}\Pr\left(\rho_o < \rho < \rho_o + \Delta\rho | \dot{\rho}_o \right)$$

$$= \frac{1}{\Delta t}\frac{f_{\rho,\dot{\rho}}\left(\rho_o, \dot{\rho}_o \right)\Delta\rho}{f_{\dot{\rho}}\left(\dot{\rho}_o \right)}$$

Averaging this over $\dot{\rho}$, we find the result $M_{\rho_o} = \int_0^{\infty} \dot{\rho} f_{\rho,\dot{\rho}}\left(\rho_o, \dot{\rho}_o \right) d\dot{\rho}$ which was first used by Rice [ ]. For our case $b_1 = 0$, the joint pdf of the amplitude and phase and corresponding derivatives is

$$f\left(\rho, \dot{\rho}, \theta, \dot{\theta} \right) = \frac{\rho^2}{4\pi^2 b_o b_2}\exp\left[-\frac{1}{2}\left(\frac{\rho^2}{b_o} + \frac{\dot{\rho}^2}{b_2} + \frac{\rho^2\dot{\theta}^2}{b_2} \right) \right].$$

Unconditioning on θ $(0 < \theta < 2\pi)$ and $\dot{\theta}\left(-\infty < \dot{\theta} < \infty \right)$ shows that, for Rayleigh-fading, $\dot{\rho}$ is zero-mean Gaussian with variance $b_2 = 2\pi^2 f_D^2 b_o$ independent of ρ. The fade margin η is defined as the ratio of the local mean power b_o and the power $\frac{1}{2}\rho_o^2$ corresponding to the threshold ρ_o, thus $\eta = \frac{2b_o}{\rho_o^2}$ and

$$M = \frac{\sqrt{2\pi} f_D}{\sqrt{\eta}} 'e^{-1/\eta}.$$

More in general, for Rician fading, the level crossing rate becomes [Rice, ‘45]

$$M_{\rho_o} = \sqrt{\pi} f_D \sqrt{b_o} f_\rho(\rho_o)$$

where b_n is the nth moment of the Doppler spectrum of the scattered power only.

<u>Exercise</u> Assume Rician fading with $k \gg 0$, so $\rho \approx c_o + I$. Show that $M_{\rho_o} \approx \frac{1}{2\pi}\sqrt{\frac{b_2}{b_o}}\exp\left\{-\frac{(\rho - c_o)^2}{2b_o}\right\}$.

In interference-limited nets, the Rayleigh-fading wanted signal often experiences interference from multiple, say n, i.i.d. Rayleigh-fading other signals. The rate of crossing a C/I-threshold z is addressed. The local fade-margin η is $\eta = \frac{p_o}{z\bar{p}_t}$ where p_t is the joint local-mean interference power $p_t = p_1 + p_2 + \ldots + p_n$. Assuming incoherent (power) cummulatiion of interference, the joint interfering signal has a Nakagami envelope ρ_t, with $\rho_t^2 = \sum_{i=1}^{n} \rho_i^2$. Given the instantaneous amplitude $\rho_1, \rho_2, \ldots \rho_n$, the derivative $\dot{\rho}_t$ is Gaussian with

$$\dot{\rho}_t = \frac{\sum_{i=1} \rho_i \dot{\rho}_i}{\rho_t}$$

If all interfering signals have the same Doppler spectrum, the variance of $\dot{\rho}_t$ is $\sigma_t^2 = 2\pi^2 f_D^2 b_o$. We now express the pdf of the signal-to-interference amplitude ratio $y\left(y^2 = \frac{\rho_o^2}{\rho_t^2}\right)$ and its derivative $\dot{y}$ in terms of the mutually independent pdfs of ρ_o, $\dot{\rho}_o$, ρ_t, and $\dot{\rho}_t$.

Since $\rho_o = \rho_t y$, we find $\dot{\rho}_o = \dot{y}\rho_t + \dot{\rho}_t y$. So,

$$f_{y,\dot{y}}\left(y_o, \dot{y}_o\right) = \int_o^\infty \int_o^\infty \rho_t \underbrace{f_{\rho_o}(\rho_t y)}_{\text{Rayleigh}} \underbrace{f_{\dot{\rho}_o}\left(\dot{y}\rho_t + \dot{\rho}_t y\right)}_{\text{Gaussian}} \underbrace{f_{\rho_t}(\rho_t)}_{\text{Nakagami}} \underbrace{f_{\dot{\rho}_t}\left(\dot{\rho}_t\right)}_{\text{Gaussian}} d\dot{\rho}_t d\rho_t$$

After some algebraic manipulations, one finds the threshold crossing rate

$$M = \sqrt{2\pi} f_D \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{n}\Gamma(n)} \sqrt{\eta} \left(1 + \frac{1}{n\eta}\right)^{-n}$$

where $\Gamma(n) = (n-1)!$ is the gamma function. The factor

$$\chi = \frac{\sqrt{\eta}\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)} \approx 1 + \frac{1}{8n} + \dots$$

varies between $\chi = \frac{2}{\sqrt{\pi}} \approx 1.13$ for $n = 1$ and $\chi \downarrow 1$ for $n \to \infty$.

Exercise Show that for $n \to \infty$, the level crossing rate ( ) is recovered.

Outage probability

An RF signal outage is the event that the signal-to-joint-interference ratio drops below minimum required threshold during a short-term observation window *T*. The duration *T* is chosen such that multiple interfering signals add incoherently, i.e., *T* is much larger that the coherence time of the modulation. Also *T* is small compared to the effects of fading $(Tf_D \ll 1)$. The probability that the C/I-ratio is above the threshold *z* is

$$'\mathrm{P}\left(\frac{p_o}{p_t} > z\right) = \int_{o-}^{\infty} \int_{zx}^{\infty} f_{p_o}(y) f_{p_t}(x)\, dy dx$$

$$= \int_{o}^{\infty} F_{p_o}(zx) f_{p_t}(x)\, dx$$

For a Rayleigh-fading wanted signal, the (cumulative) distribution is the exponential function $\exp\left(-\frac{xz}{\bar{p}_o}\right)$. So, the expression can be interpreted as the Laplace Transform of the pdf of joint interference power. For *n* i.i.d. incoherently cumulating Rayleigh-fading signals each with local-mean power p, we find

$$'P\left(\frac{p_o}{p_t} > z\right) = \left(\frac{1}{\frac{z\bar{p}}{\bar{p}_o} + 1}\right)^n = \left(1 + \frac{1}{n\eta}\right)^{-n}$$

<u>Exercise</u> Study the special cases $n = 1$ and $n = \infty$. Explain why the distribution of p_o is recovered for $n \to \infty$. For decreasing fade margins $\frac{1}{\eta} \to \infty$, the probability of successful reception vanishes slowly if $n = 1$ but rapidly if $n \to \infty$. Why?

Average (non-) fade duration

The probability of a signal outage (C/I < z) should be equal to the threshold crossing rate multiplied by the average duration of a fade. Hence, for a wanted Rayleigh-fading signal in the presence of Nakagami interference, the average nonfade duration $\bar{\tau}_{MF}$ is

$$\bar{\tau}_{MF} = \frac{'P\left(\frac{p_o}{p_t} > z\right)}{M} = \frac{1}{\sqrt{2\pi} f_D} \sqrt{\eta} \frac{\sqrt{\eta}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}$$

and the average fade duration is

$$\bar{\tau}_F = \frac{1}{\sqrt{2\pi} f_D} \sqrt{\eta} \left[\left(1 + \frac{1}{n\eta}\right)^n - 1\right] \frac{\sqrt{\eta}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}$$

<u>Exercise</u> Show that for the event of a noise-limited channel with minimum required signal power p_m, $\bar{\tau}_{NF} = \frac{\sqrt{\eta}}{\sqrt{2\pi} f_D}$ and $\bar{\tau}_F = [\exp(\eta) - 1] \frac{\sqrt{\eta}}{\sqrt{2\pi} f_D}$ with $\eta = \frac{p_o}{p_m}$.