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Properties of the signal envelope

Before we can derive properties of the Rayleigh-fading envelope ρ, with $\rho^2 = I^2 + Q^2$, we need to express correlation functions of the inphase and quadrature components $I(t)$ and $Q(t)$. Rice derived the autocorrelation $g(\tau)$ defined as

$$g(\tau) = \mathrm{E}\, I(t)\, I(t+\tau) = \mathrm{E}\, Q(t)\, Q(t+\tau)$$

$$= \int_{f_C - f_D}^{f_C + f_D} S(f) \cos 2\pi (f - f_c)\, \tau df$$

with $S(f)$ the power spectral density of the RF-signal. In the special case of uniform angles of arrival, the cross correlation $h(\tau)$, with

$$h(\tau) = \mathrm{E}\, I(t)\, Q(t+\tau) = -\mathrm{E}\, Q(t)\, I(t+\tau)$$

$$= \int_{f_C - f_D}^{f_C + f_D} S(f) \sin 2\pi (f - f_c)\, \tau df \qquad ,$$

is identical to zero for E_z, H_x and H_y. The behavior of the correlations at $\tau = 0$ can be obtained from the moments of the power spectrum

$$b_n = (2\pi)^n \int_{f_C - f_D}^{f_C + f_D} S(f)\, (f - f_c)^n df.$$

For E_z, one finds $b_n = 0$ for n is odd and

$$b_n = b_o (2\pi f_D)^n \frac{(n-1)!!}{n!}$$

where $(n-1)!! = 1 \cdot 3 \cdot 5 \ldots (n-1)!$ for n is even. In particular $b_o = 1.5\bar{p} = \frac{3E_o^2}{4}$ with $\bar{p}$ the local-mean power for an isotropic antenna. Hence,

$$\mathrm{E}I^2 = \mathrm{E}Q^2 = g(0) = b_o$$

$$\mathrm{E}\frac{dI}{dt}I = g'(0) = 0$$

$$\mathrm{E}\left(\frac{dI}{dt}\right)^2 = -g''(0) = b_2$$

The autocorrelation $g(\tau)$ is $g(\tau) = b_o J_o(2\pi f_D \tau)$ with J_o the Bessel function of zero order[ ].

Davenport and Root [ ] expressed the autocorrelation of the envelope in terms of a hypergeometric function, namely

$$R_\rho(\tau) = \frac{\pi}{2} b_o F\left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{g^2(\tau) + h^2(\tau)}{b_o^2}\right]$$

For $h(\tau) \equiv 0$, a first order expansion of F in terms of $g(\tau)$ gives $R_\rho(\tau) \approx \frac{\pi}{2}\left[b_o + \frac{1}{4}\frac{g^2(\tau)}{b_o^2}\right]$.

<u>Exercise</u> Verify that this approximation gives $\mathrm{E}\rho^2(0) \approx \frac{5\pi}{8} b_o$ which is very close to theoretical $2b_o$.

Removing the mean amplitude $\mathrm{E}\rho = \sqrt{\frac{\pi b_o}{2}}$, the autovariance is found as

$C_\rho(\tau) = R_\rho(\tau) - \mathrm{E}\rho$ where, for the field components

E_z: $$C_\rho(\tau) = \frac{\pi}{8} b_o J_o^2 (2\pi f_D \tau)$$

H_x: $$C_\rho(\tau) = \frac{\pi}{8} b_{oH} [J_o (2\pi f_D \tau) + J_2 (2\pi f_D \tau)]^2$$

H_y: $$C_\rho(\tau) = \frac{\pi}{8} b_{oH} [J_o (2\pi f_D \tau) - J_2 (2\pi f_D \tau)]^2$$

with b_{oH} the local-mean output power of a (magnetic) loop antenna $\left(b_{oH} = \frac{3E_o^2}{8} = \frac{b_o}{2} \right)$.

The power spectral density $S_\rho(f)$ of the envelope is found from the Fourier transform

$$S_\rho(f) = \frac{\pi}{2} b_o \int_{-\infty}^{\infty} R_\rho(\tau) e^{-j2\pi f\tau} d\tau$$

$$\approx \frac{\pi}{2} b_o \delta_\rho(f) + \frac{\pi}{\delta b_o} \int_{-\infty}^{\infty} (g^2(\tau) + h^2(\tau)) e^{-j2\pi f\tau} d\tau$$

Expressing the autocorrelation $g(\tau)$ and $h(\tau)$ of the inphase and quadrature components in terms of the power spectral density $S(f)$ of the received signal, we find [Jakes] for positive frequencies $(0 \le f \le 2f_m)$

$$S_\rho(f) = \frac{\pi}{2} b_o \delta(f) + \frac{1}{16 b_o} \int_{f_c - f_m}^{f_c + f_m - f} S(x) S(x + f) dx$$

It follows that the spectrum of the envelope contains frequencies up to twice the maximum Doppler shift. A mathematical explanation of this is the convolution in frequency domain resulting from the squaring-operation in expression ( ) for the autocorrelation of the envelope.

Exercise Draw a phasor diagram of two interfering waves. Show how the phase and amplitude of the phasor sum can change rapidly even for modest changes of the individual components.

For the electric field component E_z, the spectrum of the envelope is

$$S_\rho(f) = \frac{b_o}{8\pi f_D} K\left(\sqrt{1-\left(\frac{f}{2f_D}\right)^2}\right)$$

with K the elliptic integral of the first kind with $K(m) = \int_0^1 (1-t^2)^{-1/2}(1-mt^2)^{-1/2} dt$ [ ].

Exercise If a dominant wave arrives from angle α_o, the received signal power spectrum is $S(f) = S_o(f) + B\delta(f - f_C - f_D \cos\alpha_o)$. Find the spectrum of the resulting envelope. Explain why the new spectrum is still band limited to $2f_D$, but contains discontinuities or peaks at $f_D \pm f_D \cos\alpha_o$.

Derivatives of amplitude and phase

In a Rician-fading channel with zero line-of-sight amplitude $(c_o = 0)$, the inphase and quadrature component and their derivatives are zero-mean jointly Gaussian. The covariance matrix of I_o, Q_o, $\dot{I}_o$, and $\dot{Q}_o$ is

$$c = \begin{vmatrix} b_o & 0 & 0 & b_1 \\ 0 & b_o & -b_1 & 0 \\ 0 & -b_1 & b_2 & 0 \\ b_1 & 0 & 0 & b_2 \end{vmatrix},$$

where b_n is the nth moment of the Doppler spectrum of the scattered power. The determinant of this matrix is $(b_o b_2 - b_1^2)^2$.

Exercise Find the inverse of c and give the joint pdf of I_o, Q_o, $\dot{I}_o$, and $\dot{Q}_o$.

Rice [ ] expressed the pdf of ρ, $\dot{\rho}$, θ and $\dot{\theta}$ by transformation of random variables. For a Rician-fading signal with nonzero line-of-sight component, thus with $c_o \neq 0$, one can write

$$I_o = \rho\cos\theta - c_o$$

$$Q_o = \rho\sin\theta$$

$$\dot{I}_o = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta$$

$$\text{and}\,\dot{Q}_o = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta \, .$$

Hence, $dI_o dQ_o d\dot{I}_o d\dot{Q}_o = \rho^2 d\rho d\dot{\rho} d\theta d\dot{\theta}$. After some algebraic operation, this leads to

$$f(\rho, \dot{\rho}, \theta, \dot{\theta}) = \frac{\rho^2}{4\pi(b_o b_2 - b^2{}_1)} \exp - \{ \frac{1}{2b_o b_2 - 2b^2{}_1} [b_2(\rho^2 - 2c_o\rho\cos\theta$$

$$+ Q^2) + b_o(\dot{\rho}^2 + \rho^2\dot{\theta}^2) - 2b_1\rho^2\dot{\theta} + 2b_1 c_o(\dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta)] \} \, .$$

This general result is relevant to a number of more specific properties to be derived later.

<u>Exercise</u> Find $f(\rho, \dot{\rho})$ for Rayleigh-fading $(c_o = 0)$. Give an intuitive explanation why $\dot{\rho}$ is normal. Find the variance of $\dot{\rho}$.

Threshold crossing rate

The level crossing rate M_{ρ_o} is defined as the expected number of times that the envelope ρ crosses in positive direction a particular level ρ_o during one second. Given a particular derivative $\dot{\rho}_o\,(\dot{\rho}_o > 0)$ of the envelope, the duration Δt of a transition through the range $\rho_o \le \rho \le \rho_o + \Delta\rho$ is $\Delta t = \frac{\Delta\rho}{\dot{\rho}_o}$. The conditional expectation of the number of crossings per second is

$$\mathrm{E}\left[M_{\rho_o}\middle|\dot{\rho}_o\right] = \frac{1}{\Delta t}\Pr(\,(\rho_o < \rho < \rho_o + \Delta\rho\,|\,\dot{\rho}_o)$$

$$= \frac{1}{\Delta t}\frac{f_{\rho,\,\dot{\rho}}\,(\rho_o,\,\dot{\rho}_o)\,\Delta\rho}{f_{\dot{\rho}}\,(\dot{\rho}_o)}$$

Averaging this over $\dot{\rho}$, we find the result $M_{\rho_o} = \int_0^{\infty} \dot{\rho} f_{\rho,\,\dot{\rho}}\,(\rho_o,\,\dot{\rho}_o)\,d\dot{\rho}$ which was first used by Rice [ ]. For our case $b_1 = 0$, the joint pdf of the amplitude and phase and corresponding derivatives is

$$f(\rho,\,\dot{\rho},\,\theta,\,\dot{\theta}) = \frac{\rho^2}{4\pi^2 b_o b_2}\exp\left[-\frac{1}{2}\left(\frac{\rho^2}{b_o} + \frac{\dot{\rho}^2}{b_2} + \frac{\rho^2\dot{\theta}^2}{b_2}\right)\right].$$

Unconditioning on $\theta\;(0 < \theta < 2\pi)$ and $\dot{\theta}\;(-\infty < \dot{\theta} < \infty)$ shows that, for Rayleigh-fading, $\dot{\rho}$ is zero-mean Gaussian with variance $b_2 = 2\pi^2 f_D^2 b_o$ independent of ρ. The fade margin η is defined as the ratio of the local mean power b_o and the power $\frac{1}{2}\rho_o^2$ corresponding to the threshold ρ_o, thus $\eta = \frac{2b_o}{\rho_o^2}$ and

$$M = \frac{\sqrt{2\pi} f_D}{\sqrt{\eta}} e^{-1/\eta}.$$

More in general, for Rician fading, the level crossing rate becomes [Rice, '45]

$$M_{\rho_o} = \sqrt{\pi} f_D \sqrt{b_o} f_{\rho}\,(\rho_o)$$

where b_n is the nth moment of the Doppler spectrum of the scattered power only.

Exercise Assume Rician fading with $k \gg 0$, so $\rho \approx c_o + I$. Show that $M_{\rho_o} \approx \frac{1}{2\pi}\sqrt{\frac{b_2}{b_o}}\exp\left\{-\frac{(\rho - c_o)^2}{2b_o}\right\}$.

In interference-limited nets, the Rayleigh-fading wanted signal often experiences interference from multiple, say n, i.i.d. Rayleigh-fading other signals. The rate of crossing a C/I-threshold z is addressed. The local fade-margin η is $\eta = \frac{\overline{p_o}}{z\bar{p}_t}$ where $\bar{p}_t$ is the joint local-mean interference power $\bar{p}_t = \bar{p}_1 + \bar{p}_2 + \ldots + \bar{p}_n$. Assuming incoherent (power) cummulatiion of interference, the joint interfering signal has a Nakagami envelope ρ_t, with $\rho_t^2 = \sum_{i=1}^{n} \rho_i^2$. Given the instantaneous amplitude $\rho_1, \rho_2, \ldots \rho_n$, the derivative $\dot{\rho}_t$ is Gaussian with

$$\dot{\rho}_t = \frac{\sum_{i=1}^{n} \rho_i \dot{\rho}_i}{\rho_t}$$

If all interfering signals have the same Doppler spectrum, the variance of $\dot{\rho}_t$ is $\sigma_t^2 = 2\pi^2 f_D^2 b_o$.

We now express the pdf of the signal-to-interference amplitude ratio $y\left(y^2 = \frac{\rho_o^2}{\rho_t^2}\right)$ and its derivative $\dot{y}$ in terms of the mutually independent pdfs of ρ_o, $\dot{\rho}_o$, ρ_t, and $\dot{\rho}_t$.

Since $\rho_o = \rho_t y$, we find $\dot{\rho}_o = \dot{y}\rho_t + \dot{\rho}_t y$. So,

$$f_{y,\dot{y}}(y_o, \dot{y}_o) = \int_o^\infty \int_o^\infty \rho_t \underbrace{f_{\rho_o}(\rho_t y)}_{\text{Rayleigh}} \underbrace{f_{\dot{\rho}_o}(\dot{y}\rho_t + \dot{\rho}_t y)}_{\text{Gaussian}} \underbrace{f_{\rho_t}(\rho_t)}_{\text{Nakagami}} \underbrace{f_{\dot{\rho}_t}(\dot{\rho}_t)}_{\text{Gaussian}}\, d\dot{\rho}_t d\rho_t$$

After some algebraic manipulations, one finds the threshold crossing rate

$$M = \sqrt{2\pi} f_D \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{n}\,\Gamma(n)} \sqrt{\eta}\left(1 + \frac{1}{n\eta}\right)^{-n}$$

where $\Gamma(n) = (n-1)!$ is the gamma function. The factor

$$\chi = \frac{\sqrt{\eta}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)} \approx 1 + \frac{1}{8n} + \ldots$$

varies between $\chi = \frac{2}{\sqrt{\pi}} \approx 1.13$ for $n = 1$ and $\chi \downarrow 1$ for $n \to \infty$.

Exercise Show that for $n \to \infty$, the level crossing rate ( ) is recovered.

Outage probability

An RF signal outage is the event that the signal-to-joint-interference ratio drops below minimum required threshold during a short-term observation window T. The duration T is chosen such that multiple interfering signals add incoherently, i.e., T is much larger that the coherence time of the modulation. Also T is small compared to the effects of fading $(Tf_D \ll 1)$. The probability that the C/I-ratio is above the threshold z is

$$\mathrm{P}\left(\frac{p_o}{p_t} > z\right) = \int_{o-}^{\infty} \int_{zx}^{\infty} f_{p_o}(y) f_{p_t}(x)\, dy dx$$

$$= \int_{o}^{\infty} F_{p_o}(zx) f_{p_t}(x)\, dx$$

For a Rayleigh-fading wanted signal, the (cumulative) distribution is the exponential function $\exp\left(-\frac{xz}{\bar{\bar{p}}_o}\right)$. So, the expression can be interpreted as the Laplace Transform of the pdf of joint interference power. For n i.i.d. incoherently cumulating Rayleigh-fading signals each with local-mean power $\bar{p}$, we find

$$\mathrm{P}\left(\frac{p_o}{p_t} > z\right) = \left(\frac{1}{\frac{z\bar{p}}{\bar{p}_o} + 1}\right)^n = \left(1 + \frac{1}{n\eta}\right)^{-n}$$

Exercise Study the special cases $n = 1$ and $n = \infty$. Explain why the distribution of p_o is recovered for $n \to \infty$. For decreasing fade margins $\frac{1}{\eta} \to \infty$, the probability of successful reception vanishes slowly if $n = 1$ but rapidly if $n \to \infty$. Why?

Average (non-) fade duration

The probability of a signal outage (C/I < z) should be equal to the threshold crossing rate multiplied by the average duration of a fade. Hence, for a wanted Rayleigh-fading signal in the presence of Nakagami interference, the average nonfade duration $\bar{\tau}_{MF}$ is

$$\bar{\tau}_{MF} = \frac{\mathrm{P}\left(\frac{p_o}{p_t} > z\right)}{M} = \frac{1}{\sqrt{2\pi} f_D}\sqrt{\eta}\frac{\sqrt{\eta}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}$$

and the average fade duration is

$$\bar{\tau}_F = \frac{1}{\sqrt{2\pi} f_D}\sqrt{\eta}\left[\left(1 + \frac{1}{n\eta}\right)^n - 1\right]\frac{\sqrt{\eta}\,\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}$$

Exercise Show that for the event of a noise-limited channel with minimum required signal power p_m, $\bar{\tau}_{NF} = \frac{\sqrt{\eta}}{\sqrt{2\pi} f_D}$ and $\bar{\tau}_F = [\exp(\eta) - 1]\frac{\sqrt{\eta}}{\sqrt{2\pi} f_D}$ with $\eta = \frac{\bar{p}_o}{p_m}$.