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Multipath fading

If the mobile antenna receives a large number, say N, reflected and scattered waves, the instantaneous received power becomes a random variable, dependent on the location of the antenna. Initially, we address the case of an unmodulated carrier, thus the transmitted signal has the form $v(t) = \cos(\omega_c t + \psi)$. Let the nth wave with amplitude c_n and phase ϕ_n arrive from an angle α_n relative to the direction of the motion of the antenna. The Doppler shift Δf_n of this wave is

$$\Delta f_n = \frac{v}{\lambda} \cos\alpha_n,$$

where v is the speed of the antenna. The received signal $r(t)$ can be expressed as

$$r(t) = \sum_{n=1}^{N} c_n \cos(2\pi f_c t + \psi + \phi_n + 2\pi\Delta f_n t)$$

An inphase-quadrature representation of the form $r(t) = I(t)\cos\omega_c t - Q(t)\sin\omega_c t$ can be found with

$$I(t) = \sum_{n=1}^{N} c_n \cos\left(\frac{2\pi v f_c t}{c}\cos\alpha_n + \psi + \phi_n\right)$$

and

$$Q(t) = \sum_{n=1}^{N} c_n \sin\left(\frac{2\pi v f_c t}{c}\cos\alpha_n + \psi + \phi_n\right).$$

PDF of signal amplitude

In order to obtain the pdf of the signal amplitude ρ we observe the random processes $I(t)$ and $Q(t)$ at one particular instant t_o. If $N \to \infty$ and the terms in the summings are i.i.d., the central limit theorem says that $I(t_o)$ and $Q(t_o)$ are (zero-mean) Gaussian. The received signal $r(t) = \rho(t)\cos(2\pi f_c t + \theta(t))$ has a Rayleigh amplitude $\rho(t)$, with $\rho(t) = \sqrt{I^2(t) + Q^2(t)}$ and a uniform phase $\theta(t)$ with $\theta(t) \in (0, 2\pi)$. The pdf of $\rho(t)$ is

$$f_\rho(\rho) = \frac{\rho}{\sigma} exp\left(-\frac{\rho^2}{2\sigma^2}\right)$$

where σ^2 is the variance of $I(t)$ and $Q(t)$.

<u>Exercise</u> Let $z = x + jy$ with x and y i.i.d. Gaussian with zero mean and variance σ^2. Show that $\rho = |z|$ is Rayleigh, with $E\rho = \sqrt{\frac{\pi}{2}}\sigma$ and $E\rho^2 = 2\sigma^2$.

Simulations have shown that the Rayleigh pdf appropriately describes the fading of the amplitude if $N > 6$. Measurements over non-line-of-sight paths at UHF frequencies in urban environments confirmed the accuracy of the Rayleigh pdf.

The instantaneous power p, with $p = \frac{1}{2}\rho^2$,thus, averaged over one RF-cycle, has the exponential pdf

$$f_p(p) = f_\rho(\sqrt{2\rho})\left|\frac{d\rho}{dp}\right| = \frac{1}{\sigma^2} exp\left(-\frac{p}{\sigma^2}\right)$$

where $\sigma^2 = Ep = \bar{p}$ is the local-mean power.

<u>Exercise</u> Find the cumulative distribution of ρ. In a cellular voice channel, a signal outage occurs if the instantaneous signal-to-noise ratio $\frac{\rho}{N_\rho}$ is less than $z = 10$ (10 dB).

Find the required local-mean power to ensure an outage probability of less than 1%. Find the corresponding fade margin, defined as the excess power above the threshold zN_p .

If the set of reflected waves are dominated by one strong component with amplitude c_o, Rician fading is a more appropriate model. Without lack of generality we take $\phi_o = 0$, so the in-phase and quadrature component I and Q have jointly Gaussian pdf

$$f_{I,Q}(I,Q) = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-(I-c_o)^2-Q^2}{2\sigma^2}\right).$$

The amplitude $\rho = \sqrt{I^2+Q^2}$ and the phase $\theta = \arctan\left(\frac{Q}{I+c_0}\right)$ have the joint pdf

$$f_{\rho,\theta}(\rho,\theta) = \frac{\rho}{2\pi\sigma^2}\exp\left(-\frac{\rho^2-2c_o\rho\cos\theta+c_o^2}{2\sigma^2}\right)$$

with σ^2 the local-mean scattered power and $\frac{1}{2}c_o^2$ the power of the dominant component. The pdf of ρ is found from the integral

$$f_\rho(\rho) = \int_{-\pi}^{\pi} f_{\rho,\theta}(\rho,\theta)\,d\theta$$

$$= \frac{\rho}{\sigma^2}\exp\left(-\frac{\rho^2+c_o^2}{2\sigma^2}\right)I_o\left(\frac{c_o\rho}{\sigma^2}\right),$$

where I_o is the modified Bessel function of the first kind and zero order, defined as

$$I_o(x) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}d\theta$$

Exercise Show that the Rician pdf of p turns into a Rayleigh pdf for $c_o \to 0$ and into a Gaussian pdf for $c_o \gg \sigma$. Use the properties of $I_o(\bullet)$ as summarized in Mathematical handbooks, e.g. by Abramowitz.

Expressed in terms of the local-mean power $\bar{p}$ with $\bar{p} = \frac{1}{2}c_o^2 + \sigma^2$ and the Rician K-factor with $K = \frac{c_o^2}{2\sigma^2}$, the pdf of the signal amplitude becomes

$$f_\rho(\rho) = (1+K)\, e^{-K}\frac{\rho}{\bar{p}}\exp\left(-\frac{1+K}{2\bar{p}}\rho^2\right) I_o\left(\sqrt{\frac{2K(1+K)}{\bar{p}}}\rho\right)$$

Exercise Show that for a large local-mean signal-to-noise ratio $\bar{p} \gg p_n$, the probability that the instantaneous power p drops below a noise threshold p_n tends to $(K+1)\, e^{-K}\frac{\bar{p}}{p_n}$.

Experimental results showed that other pdfs can also accurately model the pdf of the received signal amplitude. An interesting property is that the incoherent (power) sum of m i.i.d. Rayleigh fading signals has a gamma distribution power $p_t\ (p_t = p_1 + p_2 \ldots + p_n)$ since

$$f_{p_t}(p_t) = \bigotimes_{i=1}^{m} \frac{1}{\bar{p}}\exp\left(-\frac{p_i}{\bar{p}}\right) = \frac{1}{\bar{p}\Gamma(m)}\left(\frac{p_t}{\bar{p}}\right)^m \exp\left(-\frac{p}{\bar{p}}\right)$$

where $\otimes$ denotes the convolution operation and $\Gamma(m)$ is the gamma function, with $\Gamma(m+1)=m!$

The amplitude ρ associated with $p\left(p = \frac{1}{2}\rho^2\right)$ then has the Nakagami m-pdf

$$f_\rho(\rho) = \frac{\rho^{2m-1}}{\Gamma(m)\, 2^{m-1}\bar{p}^m} exp\left(-\frac{\rho^2}{2\bar{p}}\right)$$

Note, however, that the incoherent sum of m signals is not a constant-envelope signal. The Nakagami pdf has been proposed as a good experimental approximation of the pdf of the amplitude of a single fading signal both for UHF mobil communication and for HF ionospheric communication. It can also approximate the Rician pdf and the log-normal pdf for small σ_s, except in the extreme tails. Special cases occur for shape factors of $m = \frac{1}{2}$, 1 and ∞ For $m = \frac{1}{2}$, a one-sided Gaussian pdf of the amplitude $(p > 0)$ is obtained. For $m = 1$, a Rayleigh pdf is recovered and for $m \to \infty, f_\rho(\rho) \to \delta(\rho - \sqrt{2m\bar{p}})$. This agrees with the fact that the incoherent sum of many independent signals behaves as a band-limited Gaussian signal. The power of such a signal (averaged over a time interval T which is substantially larger than the coherence time of the signal phases but shorter than the coherence time of any channel fading) is constant and equal to $m\bar{p}$.

RF power spectrum

We assume that $N \to \infty$ and that the angle of arrival α has the (continous) uniform pdf $f_\alpha(\alpha) = \frac{1}{2\pi}$. For frequencies in the band $f_c\left(1 - \frac{v}{c}\right) < f_o < f_c\left(1 + \frac{v}{c}\right)$, the power density spectrum of the received signal can be found from

$$S(f_o) = \bar{p}\,[f_\alpha(\alpha)\,G_R(\alpha) + f_\alpha(-\alpha)\,G(-\alpha)\,]\left|\frac{d\alpha}{df}\right|_{f_o}$$

where $\bar{p}$ is the local-mean power received by an isotropic antenna and $G_R(\alpha)$ is the antenna gain in direction α. The relation between f_o and α is $f_o = f_c\left(1 + \frac{v}{c}\cos\alpha\right)$, so

$\left|\frac{d\alpha}{d_f}\right|_{f_o} = \frac{1}{\sqrt{(f_o - f_c)^2 - f_D^2}}$, where f_D is the maximum Doppler shift $f_D = \frac{vf_c}{c}$. An electrical dipole antenna receives the electric field in vertical direction E_z with $G(\alpha) = 1.5$. So, if an unmodulated carrier is transmitted , the received spectrum is

$$S_{E_z}(f) = \frac{3}{2\pi}\frac{\bar{p}}{\sqrt{f_D^2 - (f - f_c)^2}}$$

Similiarly, one can derive the spectrum of a magnetic loop in the direction of v, receiving H_y, or perpendicular to v, receiving H_x, with antenna gains $G(\alpha) = \frac{3}{2}(\sin^2\alpha)$ and $\frac{3}{2}\cos^2\alpha$, respectively.

<u>Exercise</u> Find an expression for the spectrum $S_{E_z}(f)$ for Rician fading with K = 10, given the angle of arrival of the dominant wave α_o .